Pricing surplus server capacity for mean waiting time sensitive customers

A proof of conjecture

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Joint work with N. Hemachandra

Outline

- Joint pricing and scheduling problem
- Conjecture
- Outline of the proof

Motivation

- Firm can 'lease its facilities' to new customers without affecting the service level of inhouse customers.

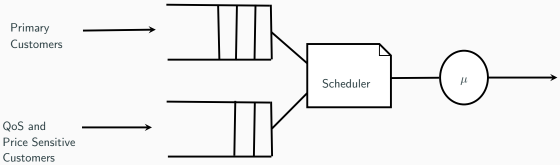

Figure 1: Model abstraction

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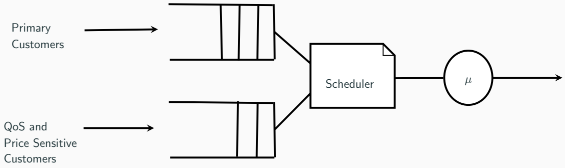
- Firm can 'lease its facilities' to new customers without affecting the service level of inhouse customers.
- Firms could be a container depot, a mobile service provider, a large manufacturing plant, etc.



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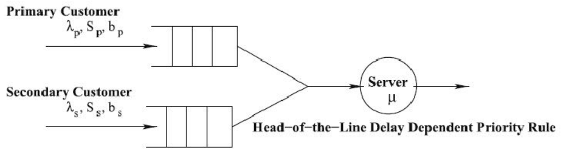

- Model to price server's surplus capacity in M/G/1 queue.

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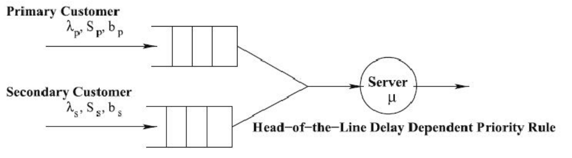

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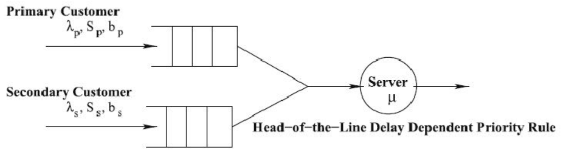

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- Primary are the existing customers and their mean waiting time is promised below S_p.

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Delay dependent priority scheduling scheme[2]

Prioritize jobs based on the following instantaneous priority:

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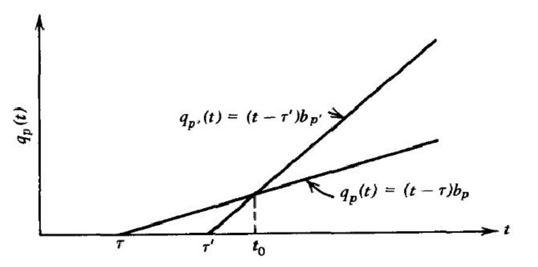

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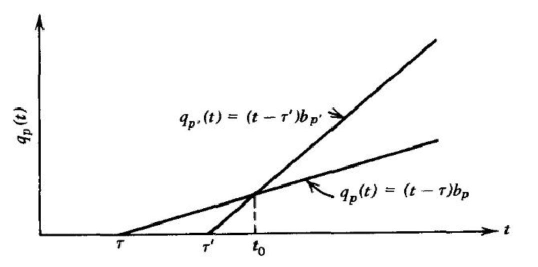

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Ties (at t_0) are broken according to FCFS

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Notations:

λ_p	Arrival rate for primary class customers
λ_s	Arrival rate of secondary customers
S_p	Promised mean waiting time of primary class customers
S_s	Promised mean waiting time of secondary class customers
μ	Mean service rate of server
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- Decision variables:
  - Unit admission price (θ), service level for new class (S_s), scheduling parameter (β) and arrival rate for secondary class (λ_s)
- Objective is to maximize total revenue.

Optimization problem P0

$$\textit{Maximize } \theta\lambda_s \tag{1}$$

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$$\lambda_s, \theta, S_s, \beta \geq 0 \tag{6}$$

- Constraint (3) and (5) will be binding.
- P0 is decomposed in P1 (with finite β) and P2 (with infinite β).

Optimization problem P1 and P2

$$\textbf{P1:} \max_{\lambda_s,\beta} \frac{1}{b}\left(a\lambda_s - \lambda_s^2 - c\lambda_s W_s(\lambda_s,\beta)\right) \tag{7}$$
$$W_p(\lambda_s,\beta) \le S_p \tag{8}$$
$$\lambda_s \le \mu - \lambda_p \tag{9}$$
$$\lambda_s, \beta \ge 0 \tag{10}$$

$$\textbf{P2:} \max_{\lambda_s} \frac{1}{b}[a\lambda_s - \lambda_s^2 - c\lambda_s \tilde{W}_s(\lambda_s)] \tag{11}$$
$$\tilde{W}_p(\lambda_s) \le S_p, \tag{12}$$
$$\lambda_s \le \mu - \lambda_p, \tag{13}$$
$$\lambda_s \ge 0. \tag{14}$$

- Solution by using KKT conditions

Solution of problem P0

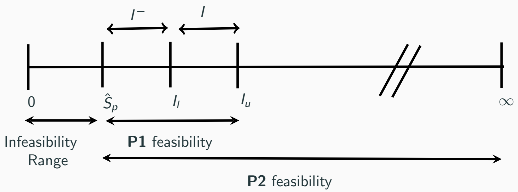
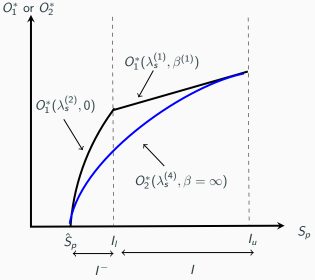


Figure 3: Range of S_p with optimal solutions coming from problem P1 and P2

- Search for global optima
- Comparison of optimal objectives of problem P1, O_1^*, and problem P2, O_2^*, in service level range $I \cup I^-$

Comparison of objectives



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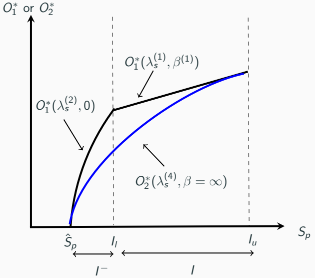

Conjecture[3]**.** *For $S_p \in I^-$, the optimal solution of P0 is given by optimal solution of P1.*

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- A finite step algorithm[4] for optimal solution.
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- Sufficient conditions[5] for conjecture to hold.

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Main result

Theorem

Optimal solution for optimization problem P_2 i.e. O_2^* is increasing concave in interval $I^- \cup I$ while O_1^* is increasing concave in I^- and linearly increasing in I

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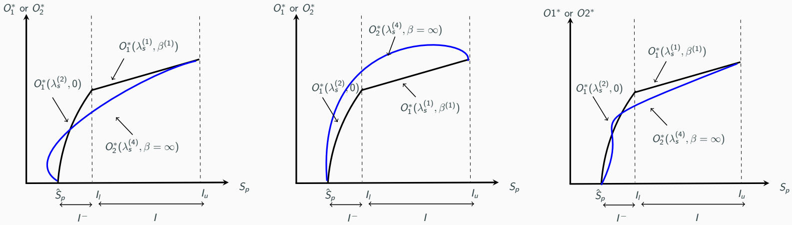
- $\lambda_s^{(4)}$ is an increasing function of S_p.

A proof by contradiction



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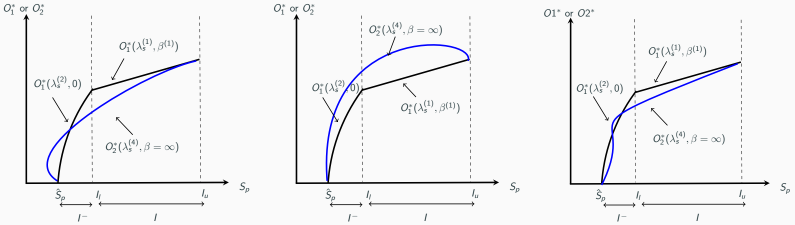

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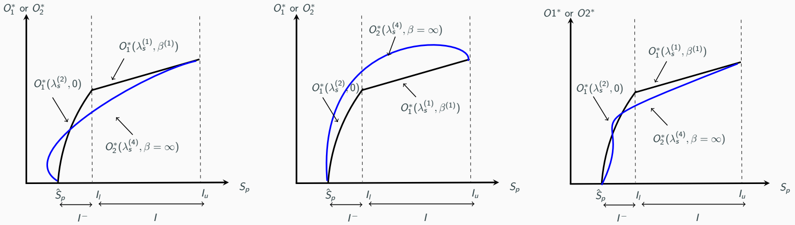

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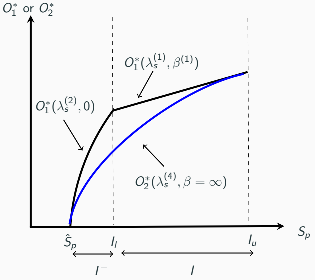
- Contradiction from concavity of O_2^*.

Possible way for O_1^* and O_2^*



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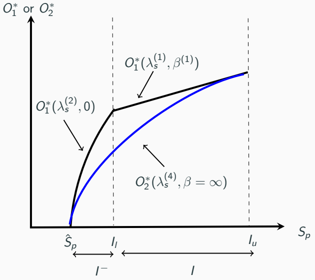

- The proof of conjecture
- Implications: Validation of finite step algorithm

Thank you!

http://manugupta-or.github.io